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A DUALITY THEOREM IN PRODUCTION CORRESPONDENCES.(U)

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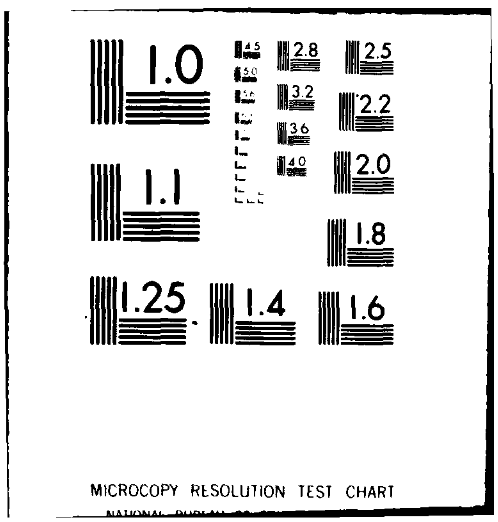
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by
KING-TIM MAK

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ABSTRACT

A generalized cost function is introduced in this report. Using this notion of cost, a theorem on the duality between the cost and distance functions of production correspondences is derived without the usual assumptions of strong-disposability of inputs nor the convexity of input sets.

A DUALITY THEOREM IN PRODUCTION CORRESPONDENCES

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King-Tim Mak

1. INTRODUCTION

Since its first rigorous treatment by Shephard [1953], the duality between the cost and production functions has been extensively investigated (for a survey, see Diewert [1978]). The corresponding duality theory for production correspondences was investigated by Jacobsen [1970] and Shephard [1970]. Almost all the duality results were derived under the assumptions of strong-disposability of inputs and convexity of the input level sets. However, as indicated in the study of semi-homogenous production structures (Shephard [1974]) and the laws of return (Shephard and Färe [1974]), the assumption of strong-disposability and convexity is not necessary to obtain meaningful results. Thus it appears to be of interest to investigate the duality between the cost and production structures under only the weak axioms of Shephard [1974]. This will be the aim of this report.

For completeness sake, Shephard's weak axioms of production will be given in the remainder of this section. Section 2 states the mathematical concepts and results needed later on. In Section 3, a generalized notion of cost is introduced and is then used to derive a theorem on the duality between the cost and distance functions of production correspondences.

(1.1) Shephard's Weak Axioms on Production

A production technology transforming inputs $x \in \mathbb{R}^n_+$ into net outputs $u \in \mathbb{R}^m_+$ is modelled by an input correspondence $L : \mathbb{R}^m_+ \to 2^{\mathbb{R}^n_+}$ where the input level set $L(u)$ denotes all input vectors which may yield output u. The input correspondence L satisfies the following properties:(1)

L.1 $L(0) = \mathbb{R}^n_+$, and $0 \notin L(u)$ for $u \geq 0$.

L.2 For $\{||u^k||\} \to +\infty$, $\bigcap_{k=1}^{\infty} L(u^k)$ is empty.

L.3 If $x \in L(u)$, $\lambda \cdot x \in L(u)$ for $\lambda \in [1,+\infty)$.

L.4 If $x \geq 0$ and $(\bar{\lambda} \cdot x) \in L(\bar{u})$, $\bar{u} \geq 0$, for some $\bar{\lambda} \in (0,+\infty)$, the ray $\{\lambda \cdot x \mid \lambda \in [0,+\infty)\}$ intersects all input sets $L(\theta \cdot \bar{u})$ for $\theta \in [0,+\infty)$.

L.5 The graph of L is closed.

L.6 $L(\theta \cdot u) \subset L(u)$ for $\theta \in [1,+\infty)$.

(1) For a discussion of the significance of these properties, see, e.g. Shephard and Färe [1974].

2. LINEAR HOMOGENOUS FUNCTIONS

The mathematical concepts and results needed later on are stated in this section. To keep this section unclustered, proofs of propositions will be given in the Appendix.

(2.1) Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^1$ is linear homogenous if for all $(\lambda, x) \in \mathbb{R}^1_+ \times \mathbb{R}^n$, $f(\lambda \cdot x) = \lambda \cdot f(x)$.

(2.2) Definition: The norm of a linear homogenous function f is defined by $||f|| := \text{Sup}\, \{f(x)/||x|| \mid x \in \mathbb{R}^n\}$.[2] If $||f|| < +\infty$, f is said to be bounded.

Because of homogeneity, it is clear that $||f||$ may be alternatively expressed as:

$$||f|| = \sup_{||x|| \leq 1} \frac{f(x)}{||x||} = \sup_{||x||=1} \frac{f(x)}{||x||} .$$

For linear homogenous functions f and g , and scalar α , addition, $f + g$, and scalar multiplication, $\alpha \cdot f$, are defined respectively by:

$$(2.3) \qquad \begin{aligned} x &\to (f+g)(x) := f(x) + g(x) \;; \\ x &\to (\alpha \cdot f)(x) := \alpha \cdot f(x) \;. \end{aligned}$$

With (2.3), it is clear that the space of linear homogenous functions is a normed vector space with origin $f \equiv 0$.

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(2) $||\cdot||$ denotes both the norm of a function and the Euclidean norm of vectors in $\mathbb{R}^n$.

(2.4) <u>Proposition</u>: The space H of bounded <u>continuous</u> linear homogenous functions on $\mathbb{R}^n$ is a Banach space.

We shall be mainly concerned with a subclass of H, namely, the class of nonnegative bounded continuous linear homogenous functions: $H_+ := \{f \in H \mid f \geqq 0\}$.[3] The class H_+ has rather interesting geometric properties. To see this, several definitions are needed; the terminology basically follows that of Arrow and Hahn [1971]:

(2.5) <u>Definition</u>: For a set $S \subset \mathbb{R}^n$ and points $x, y \in S$, x is <u>visible</u> in S from y if $\lambda \cdot x + (1-\lambda) \cdot y \in S$ for each $\lambda \in [0,1]$.

(2.6) <u>Definition</u>: A set S in $\mathbb{R}^n$ is <u>star-shaped</u> (with center x^o) if $x^o \in S$ and every point in S is visible from x^o.

The <u>gauge function</u> $p(\cdot \mid x^o, S)$ centered at x^o for a set $S \subset \mathbb{R}^n$, $x^o \in S$, is defined for all $x \in M[S]$[4] as: $p(x \mid x^o, S) := \mathrm{Inf}\,\{\lambda \mid \lambda > 0,\ x^o + (x - x^o)/\lambda \in S\}$.

(2.7) <u>Definition</u>: A set S in $\mathbb{R}^n$ is <u>nonedged</u> if for each $x \in \partial S$[5], the point x is the only element of the intersection $\{\lambda \cdot x \mid \lambda \geq 0\} \cap \partial S$.

(2.8) <u>Proposition</u>: If f is a nonnegative bounded continuous linear homogenous function on $\mathbb{R}^n$, i.e., $f \in H_+$, then there exists a closed, nonedged and star-shaped set S (centered at 0) with

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[3] $f \geqq 0$ if $f(x) \geqq 0$ for all $x \in \mathbb{R}^n$.

[4] $M[S]$ denotes the subspace of $\mathbb{R}^n$ spanned by S.

[5] ∂S denotes the boundary of the set S.

$0 \in \text{int}(S)$[6] such that f is the gauge function of S centered at 0. Conversely, if S is a closed, nonedged and star-shaped set (centered at 0) with $0 \in \text{int}(S)$, then its gauge function centered at 0 belongs to H_+.

(2.9) Proposition: Suppose a closed set S in $\mathbb{R}^n$ satisfies (i) $0 \notin S$; and (ii) $y \in S$ and $\lambda \geqq 1$ imply $\lambda \cdot y \in S$; then for every point $x^o \notin S$, $x^o \neq 0$, there exists a bounded continuous linear homogenous function $f \in H_+$ such that $f(x^o) < f(y)$ for every $y \in S$; in fact, f may be chosen such that $0 < f(x^o) < \text{Inf}\,\{f(y) \mid y \in S\}$.

Proposition (2.8) gives the geometric property of functions in the space H_+. The "separation" proposition (2.9) is instrumental for the proof in the next section. Note that it is quite similar to the separation theorem involving linear functionals (hyperplanes) and convex sets.

[6] int(S) denotes the interior of set S.

3. GENERALIZED COST FUNCTIONS AND DUALITY

Since the technology modelled here has multiple outputs, it cannot be represented by the usual production function. However, it is convenient to have a functional representation of the production correspondence. The "distance" function is defined for this purpose. And the duality theorem to be derived in this section is given in terms of the distance function and a generalized cost function.

(3.1) Definition (Shephard [1970]): The distance function $\Psi : \mathbb{R}^m_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ for an input correspondence $L : \mathbb{R}^m_+ \to 2^{\mathbb{R}^n_+}$ is given by

$$(u,x) \to \Psi(u,x) := \begin{cases} 0 \text{ , if } L(u) = \emptyset \text{ ;} \\ 0 \text{ , if } L(u) \neq \emptyset \text{ , } \{\theta \cdot x \mid \theta \geq 0\} \cap L(u) = \emptyset \text{ ;} \\ [\operatorname{Inf} \{\lambda \mid \lambda > 0 \text{ , } \lambda \cdot x \in L(u)\}]^{-1} \text{ , if otherwise.} \end{cases}$$

The relationship between the distance function Ψ and the input correspondence L is given by the following:

(3.2) Proposition (Shephard [1970]): For all $u \in \mathbb{R}^m_+$, $L(u) = \{x \in \mathbb{R}^n_+ \mid \Psi(u,x) \geqq 1\}$.

The cost of production is usually given in terms of nonnegative price vectors. Formally, the cost function is defined as a mapping $Q : \mathbb{R}^m_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$,

$$(u,p) \to Q(u,p) := \operatorname{Inf} \{p \cdot x \mid x \in L(u)\} \text{ .}$$

The underlying assumption of the above definition is the existence of competitive markets for the inputs and that the producer is a price-taker. When the producer is a monopolist, or when competitive markets do not exist, a more general definition of "cost" is needed. However, it seems reasonable to postulate that "cost" rises proportionately with the usuage of inputs.

Hence, the following definition:

(3.3) <u>Definition</u>: The <u>generalized cost function</u> $K : \mathbb{R}^m_+ \times H_+ \to \mathbb{R}_+$ for an input correspondence $L : \mathbb{R}^m_+ \to 2^{\mathbb{R}^n_+}$ is given by:

$$(u,f) \to K(u,f) := \begin{cases} +\infty & \text{if } L(u) = \emptyset ; \\ \text{Inf } \{f(x) \mid x \in L(u)\} & \text{if } L(u) \neq \emptyset . \end{cases}$$

Note that mathematically, the cost function Q is a special case of the general definition. Furthermore, the above definition covers the situation where evaluation of input expenditure depends on the mix (the combination) of the input vectors.

Properties of the generalized cost function is given in the next proposition. They are quite similar to those of the usual cost function Q (see Shephard [1970, Proposition 22]):

(3.4) <u>Proposition</u>: The generalized cost function K has the following properties:

(K.0) If $f \equiv 0$ or $u = 0$, $K(u,f) = 0$.

(K.1) $K(u,\lambda\cdot f) = \lambda\cdot K(u,f)$, $\lambda \in \mathbb{R}^1_+$.

(K.2) $K(u,f) \geqq K(u,g)$ if $f \geqq g$.[7]

(K.3) Given $u \in \mathbb{R}^m_+$, $K(u,\cdot)$ is a concave function on H_+ .

(K.4) Given $u \in \mathbb{R}^m_+$, $K(u,\cdot)$ is a continuous function on H_+ .

(K.5) $K(\lambda\cdot u,f) \geqq K(u,f)$, $\lambda \in [1,+\infty)$.

(K.6) Given $f > 0$,[8] $K(\cdot,f)$ is l.s.c. on $\mathbb{R}^n_+$.

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[7] $f \geqq g$ iff $(f - g) \in H_+$.

[8] $f > 0$ iff $f(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

Proof:

Items K.0, K.1, K.2 and K.5 follow from the definition of K and Shephard's weak axioms on L.

To show K.3, let $f, g \in H_+$ and $\theta \in [0,1]$. Let $h := \theta \cdot f + (1-\theta) \cdot g$. Then for every $x \in L(u)$, $L(u) \neq \emptyset$, $h(x) = \theta \cdot f(x) + (1-\theta) \cdot g(x) \geqq \theta \cdot K(u,f) + (1-\theta) \cdot K(u,g)$. If $L(u) = \emptyset$, $K(u,f) = K(u,g) = K(u,h) = +\infty$. Hence $K(u,\cdot)$ is a concave function on H_+.

To show K.4, extend the definition of K to all of H. Using the argument above, it is easy to see that given $u \in \mathbb{R}^m_+$, the extended $K(u,\cdot)$ function is concave on H, hence continuous on H; consequently $K(u,\cdot)$ is continuous on the restricted domain H_+.

To show K.6, consider arbitrary $f \in H_+$, $f > 0$ and infinite sequence $\{u^k\} \subset \mathbb{R}^m_+$, $u^k \to u^o$. If $\liminf K(u^k,f) = +\infty$, then clearly $\liminf K(u^k,f) \geqq K(u^o,f)$. So suppose the sequence $\{K(u^k,f)\}$ is bounded. For arbitrary $\varepsilon > 0$, for all k there exists (by definition of $K(u^k,f)$) an input $x^k \in L(u^k)$ such that $f(x^k) < K(u^k,f) + \varepsilon$. The fact that f is continuous and $f > 0$ implies that the sequence $\{x^k\}$ is bounded. Hence there exists a convergent subsequence $\{x^j\} \subset \{x^k\}$ with $x^j \to x^o$. Then, by L.5 of Shephard's axioms, $x^o \in L(u^o)$. Clearly $K(u^o,f) \leqq f(x^o) \leqq \liminf K(u^k,f) + \varepsilon$. Letting $\varepsilon \downarrow 0$, the lower semi-continuity of $K(\cdot,f)$ is established. Q.E.D.

We are now ready to establish the duality theorem. The method of proof follows rather closely that of Shephard [1970, Proposition 44]:

(3.5) Theorem: The generalized cost function K and the distance function Ψ is related by: for all $u \in \mathbb{R}^m_+$,

$$(3.5.1)\qquad K(u,f) = \text{Inf}\ \{f(x) \mid x \in \mathbb{R}^n_+ ,\ \Psi(u,x) \geqq 1\} ,\ f \in H_+ ;$$

$$(3.5.2)\qquad \Psi(u,x) = \text{Inf}\ \{f(x) \mid f \in H_+ ,\ K(u,f) \geqq 1\} ,\ x \in \mathbb{R}^n_+ .$$

Proof:

In view of Proposition (3.2), (3.5.1) is only a restatement of the definition of K.

If $L(u) = \emptyset$, $\Psi(u,x) = 0$ for all $x \in \mathbb{R}^n_+$ and $K(u,f) = +\infty$ for all $f \in H_+$. By taking the norm $||f||$ arbitrarily small, we see that for all $x \in \mathbb{R}^n_+$, $\text{Inf}\ \{f(x) \mid f \in H_+ ,\ K(u,f) \geqq 1\} = 0$; establishing (3.5.2) for the case of u with $L(u)$ empty.

If $u = 0$, $L(u) = \mathbb{R}^n_+$. Then clearly for all $f \in H_+$, $K(u,f) = 0$ and for all $x \in \mathbb{R}^n_+$, $\Psi(u,x) = +\infty$. Thus (3.5.2) holds also for the case $u = 0$.

Now consider $u \geq 0$ with $L(u) \neq \emptyset$. Define the function $\Psi^*(u,\cdot)$ and the set $L^*(u)$ by

$$\Psi^*(u,x) := \text{Inf}\ \{f(x) \mid f \in H_+ ,\ K(u,f) \geqq 1\} ;$$

$$L^*(u) := \{x \in \mathbb{R}^n_+ \quad \Psi^*(u,x) \geqq 1\} .$$

We shall show that $L(u) \subset L^*(u)$ and $L^*(u) \subset L(u)$.

Suppose $x^o \in L(u)$. By the definition of Ψ^*, there exists for each $\varepsilon > 0$ a function $f^\varepsilon \in H_+$ and $K(u,f^\varepsilon) \geqq 1$ such that $\Psi^*(u,x^o) + \varepsilon > f^\varepsilon(x^o)$. Fix arbitrary $\theta > 0$ and let $f^o := \theta\cdot f^\varepsilon$. Then by K.1, $K(u,f^o) = \theta\cdot K(u,f^\varepsilon)$, which gives $f^\varepsilon = f^o/\theta = (K(u,f^\varepsilon)/K(u,f^o))\cdot f^o$. Thus

$$f^{\varepsilon}(x^{o}) = \left(\frac{K(u,f^{\varepsilon})}{K(u,f^{o})}\cdot f\right)(x^{o}) = \frac{K(u,f^{\varepsilon})}{K(u,f^{o})}\cdot f(x^{o}) \geqq K(u,f^{\varepsilon}) \geqq 1 \ .$$

By taking ε arbitrarily small, we have $\Psi^{*}(u,x^{o}) \geqq 1$, hence $x^{o} \in L^{*}(u)$; establishing $L(u) \subset L^{*}(u)$.

To show the converse inclusion, suppose $x^{o} \notin L(u)$, $x^{o} \neq 0$. Because of Axioms L.1, L.3 and L.5, the hypothesis of Proposition (2.9) is satisfied for the set $L(u)$. Hence there exists $f \in H_{+}$ such that $0 < f(x^{o}) < K(u,f)$.[9] Let $f^{*} := f/K(u,f)$. Then $f^{*}(x^{o}) < 1$. Now, by K.1, $K(u,f^{*}) = 1$. Hence $\Psi^{*}(u,x^{o}) \leqq f^{*}(x^{o}) < 1$, implying that $x^{o} \notin L^{*}(u)$. If $x^{o} = 0$, clearly $x^{o} \notin L(u)$ and $x^{o} \notin L^{*}(u)$. Hence $L^{*}(u) \subset L(u)$.

Finally, we shall establish that in fact $\Psi^{*}(u,x) = \Psi(u,x)$.

Using the fact that $L(u) \subset L^{*}(u)$, we shall show that for all $x \in \mathbb{R}^{n}_{+}$, $\Psi^{*}(u,x) \geqq \Psi(u,x)$. There are two cases to eliminate:

(a) $\Psi(u,x) > \Psi^{*}(u,x) = 0$; and

(b) $\Psi(u,x) > \Psi^{*}(u,x) > 0$.

For case (a), suppose $\Psi(u,x^{o}) > \Psi^{*}(u,x^{o}) = 0$. Let $\lambda := [\Psi(u,x^{o})]^{-1} > 0$. By the homogeneity of $\Psi(u,\cdot)$ (see Shephard [1970], Proposition 16), $\Psi(u,\lambda\cdot x^{o}) = 1$. Hence $\lambda\cdot x^{o} \in L(u) \subset L^{*}(u)$, implying that $\Psi^{*}(u,\lambda\cdot x^{o}) \geqq 1$. It is clear from the definition of Ψ^{*} that $\Psi^{*}(u,\lambda\cdot x^{o}) = \lambda\cdot\Psi^{*}(u,x^{o})$. Hence $\Psi^{*}(u,x^{o}) \geqq 1/\lambda > 0$, a contradiction. For case (b), suppose $\Psi(u,x^{o}) > \Psi^{*}(u,x^{o}) > 0$. Again, with $\lambda := [\Psi(u,x^{o})]^{-1}$, we argue as above to get $\Psi^{*}(u,\lambda\cdot x^{o}) = \lambda\cdot\Psi^{*}(u,x^{o}) \geqq 1 = \lambda\cdot\Psi(u,x^{o})$, or $\Psi^{*}(u,x^{o}) \geqq \Psi(u,x^{o})$, a contradiction.

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[9] That $f(x^{o}) > 0$ is clear from the proof of Proposition (2.9) given in the Appendix.

Next, using the fact that $L^*(u) \subset L(u)$, we may establish analogously that $\Psi(u,x) \geqq \Psi^*(u,x)$ and our proof is then completed. Q.E.D.

(3.6) Remark: In the definition of the generalized cost functions, we tried to limit the class of cost functions to those generated by nonnegative bounded continuous homogenous functions. However, H_+ is still a rather large class of functions. Whether there are smaller classes of cost functions which still maintain the duality between the cost and distance functions (under Shephard's weak axiom) is an open question.

APPENDIX

Proof of Proposition (2.4):

It is clear from definition that H is a normed space, so it remains to show that H is complete. Let $\{f_k\}$ be a Cauchy sequence in H. For every $x \in \mathbb{R}^n$, $\{f_k(x)\}$ is then a Cauchy sequence of real numbers which has a limit. Let the limit be denoted $f(x)$. The function $x \to f(x)$ defined this way is clearly homogenous: for $\lambda \in \mathbb{R}^1_+$, $f(\lambda x) = \lim f_k(\lambda x) = \lim \lambda \cdot f_k(x) = \lambda \cdot \lim f_k(x) = \lambda \cdot f(x)$.

Since f_k's are continuous, $\{f_k\}$ converges to f uniformly on the unit surface $\Gamma := \{x \in \mathbb{R}^n \mid ||x|| = 1\}$. Then for arbitrary $\varepsilon > 0$, there exists integer K such that if $m \geq K$, $|f(x) - f_m(x)| \leq \varepsilon$ for all $x \in \Gamma$. Thus, for a fixed $m \in K$ and all $x \in \Gamma$, $|f(x)| = |f(x) - f_m(x) + f_m(x)| \leq |f(x) - f_m(x)| + |f_m(x)| \leq (\varepsilon + |f_m|) \cdot |x|$. That is, f is a bounded homogenous function.

Finally, the continuity of f follows from the fact that: if $\{x^k\} \to x^o$, $|f(x^k) - f(x^o)| = |f(x^k) - f_m(x^k) + f_m(x^k) - f_m(x^o) + f_m(x^o) - f(x^o)| \leq |f(x^k) - f_m(x^k)| + |f_m(x^k) - f_m(x^o)| + |f_m(x^o) - f(x^o)|$ which converges to zero as m, k go to $+\infty$. Q.E.D.

Arrow and Hahn [1971] called a set S strictly star-shaped (with center x^o) if there is a relative neighborhood N of x^o such that every point of S is visible from every point of N. They showed that the gauge function (centered also at x^o) for such a set S is continuous. Since we do not pre-suppose that $0 \in \text{int}(S)$, our proof of continuity of the gauge function uses the notion of "non-edged-ness" rather than strictly star-shaped (with center 0).

Proof of Proposition (2.8):

Suppose $f \in H_+$. Consider the lower level set of f at value 1 defined by $S := \{x \in \mathbb{R}^n \mid f(x) \leqq 1\}$. We shall show that the set S is the desired set.

Firstly, $0 \in S$ since $f(0) = 0$. Next, we note that the gauge function of S (centered 0) is f itself: $p(x \mid 0,S) = \operatorname{Inf} \{\lambda \mid \lambda > 0 , x/\lambda \in S\} = \operatorname{Inf} \{\lambda \mid \lambda > 0 , f(x/\lambda) \leqq 1\} = \operatorname{Inf} \{\lambda \mid \lambda > 0 , f(x) \leqq \lambda\} = f(x)$.

That the set S is closed follows from the continuity of f . Clearly, S is star-shaped with center 0. Since if it is not so, there exists $x \in S$ and x not visible from 0. That is, exists $\theta \in (0,1)$ such that $\theta \cdot x \notin S$. This implies $f(\theta \cdot x) > 1$, contradicting the homogeneity of f .

Next, contra-positive argument is used to show that S is non-edged. Suppose there exists $x^o \in S$ and $\lambda \in \mathbb{R}^1_+$, $\lambda \cdot x^o \neq x^o$ such that the points x^o and $\lambda \cdot x^o$ both belongs to ∂S (the boundary of S). Without loss of generality, assume $\lambda > 1$. Since S is closed, $\partial S \subset S$ implying that $f(x^o) < f(\lambda \cdot x^o) \leqq 1$. Since $x^o \in \partial S$, there exists an infinite sequence $\{x^k\} \subset S^{c(*)}$ such that $x^k \to x^o$. Note that $f(x^k) > 1$ whereas $f(x^o) < 1$; resulting in the contradiction that f is not continuous at x^o .

Finally, suppose $0 \notin \operatorname{int}(S)$. Then there exists an infinite sequence $\{x^k\} \subset S^c$ with $||x^k|| \to 0$. Since $f(x^k) > 1$ for all k , $f(x^k)/||x^k|| > 1/||x^k|| \to +\infty$, implying that $||f||$ is not bounded, a contradiction.

Thus S is a closed, non-edged and star-shaped set (centered at 0) with $0 \in \operatorname{int}(S)$ whose gauge function (centered at 0) is f , establishing the first part of proposition.

$^{(*)}S^c$ denotes the complement of S .

To show the second part, let S be a closed, non-edged and star-shaped set (centered 0) with $0 \in \mathrm{int}(S)$. For simplicity, denote the gauge function of S (centered 0) as p. That p is linear homogenous and $p(0) = 0$ follows from the definition of gauge functions. So, we only have to show that p is a bounded continuous function.

Suppose p is not bounded. Then there exists an infinite sequence $\{x^k\}$ such that $\theta^k := p(x^k)/||x^k|| \to +\infty$. Since S is closed, star-shaped (centered at 0) and $0 \in \mathrm{int}(S)$, it is easy to verify that $S = \{x \in \mathbb{R}^n \mid p(x) \leqq 1\}$. By the homogeneity of p, $p(2 \cdot x^k/\theta^k \cdot ||x^k||) = 2$. Hence, for all k, $y^k := 2x^k/\theta^k \cdot ||x^k|| \notin S$. The fact that $||y^k|| \to 0$ implies 0 is not an interior point of S, a contradiction. Thus p is a bounded function.

If $S = \mathbb{R}^n$, then clearly $p \equiv 0$ and thus continuous. So, suppose $S \neq \mathbb{R}^n$. To show the continuity of p, we shall show that p is both lower and upper semi-continuous. The lower semi-continuity of p follows from the fact that the sets $\{x \mid p(x) \leqq \gamma\} = \gamma \cdot \{x \mid p(x) \leqq 1\} = \gamma \cdot S$ are closed for all $\gamma \geq 0$ and that $\{x \mid p(x) \leqq \gamma\} = \emptyset$ if $\gamma < 0$.

Now, let $\{x^k\}$ be an arbitrary infinite sequence converging to x^o. We may as well assume that for some $B > 0$, $||x^k|| \leqq B$ for all k.

Since p is bounded and not identically zero, $0 < ||p|| =: N < +\infty$. Hence for all k, $p(x^k) \leqq ||p|| \cdot ||x^k|| \leqq N \cdot B$. Then by homogeneity of p, $p(x^k/NB) \leqq 1$ implying that $(x^k/NB) \in S$ for all k. Furthermore, $(x^k/NB) \to (x^o/NB)$. For ease of notation, denote $z^k := x^k/NB$, $k = 0,1, \ldots$. Suppose $\sigma := \limsup p(z^k) > p(z^o)$. Then for arbitrary $\alpha \in (p(z^o),\sigma)$, $\alpha > 0$ and there exists an infinite subsequence $\{z^m\} \subset \{z^k\}$ with $p(z^m/\alpha) \geq 1$ and $p(z^o/\alpha) < 1$. In other words, $z^o/\alpha \in S$ while $z^m/\alpha \notin S$ for all m. The fact that $z^m/\alpha \to z^o/\alpha$ implies that z^o/α

is in the boundary ∂S . However, this is true for every α belonging to the nontrivial interval $(p(z^o),\sigma)$. Hence S is not non-edged; a contradiction. Thus $\limsup p(z^k) \leq p(z^o)$ and consequently $\limsup p(x^k) \leq p(x^o)$. Since $\{x^k\}$ was arbitrarily chosen, p is u.s.c. Q.E.D.

Proof of Proposition (2.9):

Let S satisfy conditions (i) and (ii) and $x^o \notin S$, $x^o \neq 0$. Since S is closed, there exists open balls $B_\varepsilon(x^o)$ and $B_\sigma(0)$ (radius ε and center x^o , radius σ and center 0 respectively) such that $B_\varepsilon(x^o) \cap S = B_\sigma(0) \cap S = \emptyset$.

Define the set $W := \{x \in \mathbb{R}^n \mid x = \lambda z$ for some $z \in B\ (x^o)$ and $\lambda \in [0,1]\}$. Note that $W \cap S = \emptyset$. This is because if $y \in W \cap S$ ($y \neq 0$ because of (i)), there must exist $\theta \geq 1$ such that $\theta \cdot y \in B_\varepsilon(x^o)$. But by (ii), $\theta \cdot y \in S$; contradicting the fact that $B_\varepsilon(x^o) \cap S = \emptyset$.

Now let $\bar{B}_{\sigma/2}(0)$ be the closed ball with radius $\sigma/2$ centered at 0. Let $A :=$ Closure $[W \cap \bar{B}_{\sigma/2}(0)]$. Finally, define the set $B := \text{Conv}\,[\{x^o\} \cup A] \cup \bar{B}_{\sigma/2}(0)$ whose Conv $[\cdot]$ is the convex hull of the set in the argument. Clearly $B \subset W \cup B_\varepsilon(0)$, hence $B \cap S = \emptyset$. Furthermore, it is straightforward to show that B is closed, non-edged and star-shaped (centered at 0) with $0 \in \text{int}(B)$. Also, there exists $\beta < 1$ such that for all $z \in S$, $\text{Max}\,\{\lambda \mid \lambda z\ ,\ z \in B\} < \beta$.

Then by Proposition (2.8), the gauge function p of B (centered at 0) is a nonnegative bounded continuous linear homogenous function. And $p(z) > 1$ for all $z \in S$; in fact $\text{Inf}\,\{p(z) \mid z \in S\} > 1/\beta > 1$. On the other hand, $x^o \in B$, hence $p(x^o) \leq 1$ and $p(x^o) > 0$ since $x^o \neq 0$. Q.E.D.

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